Question: Find $\mathcal{L}\{t^{\alpha}\}$

The Gamma Function: $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt$

$$\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = -e^{-t}|_0^{\infty} = 1$$

$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^{x}\, dt = \lim_{n\to\infty} \int_0^{n} e^{-t} t^{x}\, dt$$
$$= \lim_{n\to\infty} \left(-e^{-t} t^{x}|_0^{n} + \int_0^{n} x e^{-t} t^{x-1}\, dt \right)$$
$$= -e^{-t} t^{x}|_0^{\infty} + x \int_0^{\infty} e^{-t} t^{x-1}\, dt$$
$$= 0 + x\Gamma(x) = x\Gamma(x)$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-t} t^{-1/2}\, dt$$

Let $u = \sqrt{t} \;\rightarrow\; t = u^2$
Then $du = \frac{1}{2\sqrt{t}} dt \;\rightarrow\; t^{-1/2} dt = 2du$

$$= 2\int_0^{\infty} e^{-u^2}\, du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

Now back to actually finding $\mathcal{L}\{t^{\alpha}\}$: where $\alpha > -1$ and real

$$\mathcal{L}\{t^{\alpha}\} = \int_0^{\infty} e^{-st} t^{\alpha}\, dt$$

Let $u = st \;\rightarrow\; t = \frac{u}{s} \;\rightarrow\; dt = \frac{du}{s}$

$$= \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{\alpha} \frac{du}{s}$$

$$= \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-u} u^{\alpha}\, du$$

$$= \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1)$$

Need $u = st > 0$ so $s > 0$

So we get that

$$\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$